

# TOPOLOGICAL PROPERTIES OF MANIFOLDS ADMITTING A $Y^x$ -RIEMANNIAN METRIC

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**ABSTRACT.** A complete Riemannian manifold  $(M, g)$  is a  $Y_l^x$ -manifold if every unit speed geodesic  $\gamma(t)$  originating at  $\gamma(0) = x \in M$  satisfies  $\gamma(l) = x$  for  $0 \neq l \in \mathbb{R}$ . Bérard-Bergery proved that if  $(M^m, g), m > 1$  is a  $Y_l^x$ -manifold, then  $M$  is a closed manifold with finite fundamental group, and the cohomology ring  $H^*(M, \mathbb{Q})$  is generated by one element.

We say that  $(M, g)$  is a  $Y^x$ -manifold if for every  $\epsilon > 0$  there exists  $l > \epsilon$  such that for every unit speed geodesic  $\gamma(t)$  originating at  $x$ , the point  $\gamma(l)$  is  $\epsilon$ -close to  $x$ . We use Low's notion of refocussing Lorentzian space-times to show that if  $(M^m, g), m > 1$  is a  $Y^x$ -manifold, then  $M$  is a closed manifold with finite fundamental group. As a corollary we get that a Riemannian covering of a  $Y^x$ -manifold is a  $Y^x$ -manifold. Another corollary is that if  $(M^m, g), m = 2, 3$  is a  $Y^x$ -manifold, then  $(M, h)$  is a  $Y_l^x$ -manifold for some metric  $h$ .

## 1. INTRODUCTION

Throughout the paper all Riemannian manifolds are assumed to be geodesically complete, while Lorentzian manifolds (see Section 4) are not assumed to be complete unless this is explicitly stated. We will tacitly assume that a manifold  $M$  under consideration is a smooth connected manifold without boundary (not necessarily compact or oriented).

**1.1. Definition** ( $Y_l^x$ -manifolds). Let  $(M, g)$  be a Riemannian manifold,  $x$  a point in  $M$  and  $l$  a nonzero real number. We say that  $(M, g)$  is a  $Y_l^x$ -manifold if for every geodesic  $\gamma : \mathbb{R} \rightarrow M$  satisfying  $\gamma(0) = x$  and  $|\dot{\gamma}(0)| = 1$  we have  $\gamma(l) = x$ .

In other words  $(M, g)$  is a  $Y_l^x$ -manifold if each geodesic parametrized by arc length and emitted from  $x$  comes back to  $x$  at the moment  $l$ . Such manifolds attracted a lot of attention [10]. They are related to Blaschke manifolds and manifolds all of whose geodesics are closed. The following weak form of the Bott-Samelson Theorem [11, 28] was proved by Bérard-Bergery, see [5], [10, Theorem 7.37, page 192].

**1.2. Theorem** (Bérard-Bergery). *If  $(M, g)$  is a  $Y_l^x$ -Riemannian manifold of dimension at least 2, then  $M$  is a closed manifold with finite fundamental group and the cohomology ring  $H^*(M, \mathbb{Q})$  is generated by one element.*

The standard metric on  $S^1$  shows that the statement of Theorem 1.2 is false for one dimensional  $(M, g)$ .

**1.3. Remark.** Besse [10, Definitions 7.7] describes a few notions closely related to  $Y_l^x$ -manifolds. In particular,  $(M, g)$  is a  $Z^x$ -manifold if all the geodesics starting at

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$x$  come back to  $x$ . Clearly every  $Y_l^x$ -manifold is a  $Z^x$ -manifold. However according to [10, Question 7.70] it is not known if every  $Z^x$ -manifold is a  $Y_l^x$ -manifold for some nonzero  $l$ . Moreover it is not known if the length of the first return to  $x$  of a unit speed geodesic starting at  $x$  is uniformly bounded for every initial unit vector in  $T_x M$ , see [10, Question 7.71].

In this work we introduce and study the class of  $\tilde{Y}^x$ -manifolds that generalizes  $Y_l^x$ -manifolds. Remark 1.3 suggests that a priori even quite simple questions about manifolds satisfying conditions close to the definition of the  $Y_l^x$ -manifold can be hard to answer. We show however that all  $\tilde{Y}^x$ -manifolds satisfy a counterpart of the Bérard-Bergery Theorem (see Theorem 2.5).

## 2. MAIN RESULTS AND DEFINITIONS

**2.1. Definition** ( $Y^x$ -manifolds). Let  $(M, g)$  be a Riemannian manifold and  $x$  a point in  $M$ . We say that  $(M, g)$  is a  $Y^x$ -manifold if there exists  $\bar{\epsilon} > 0$  such that for every positive  $\epsilon < \bar{\epsilon}$  there exists  $l = l(\epsilon)$  with  $l > \epsilon$  such that for every unit speed geodesic  $\gamma : \mathbb{R} \rightarrow M$  originating at  $x$  at time 0, the point  $\gamma(l)$  is  $\epsilon$  close to  $x$ .

In other words, in a  $Y^x$ -manifold for every sufficiently small neighborhood of  $x$  there exists a moment of time  $l$  when all the unit speed geodesics emitted from  $x$  return back to this neighborhood (after first leaving the neighborhood).

**2.2. Definition** ( $\tilde{Y}^x$ -manifolds). Let  $(M, g)$  be a Riemannian manifold and  $x$  a point in  $M$ . We say that  $(M, g)$  is a  $\tilde{Y}^x$ -manifold if there exists  $\bar{\epsilon} > 0$  such that for every positive  $\epsilon < \bar{\epsilon}$  there exist  $l = l(\epsilon)$  with  $l > \epsilon$  and  $y = y(\epsilon)$  such that for every unit speed geodesic  $\gamma : \mathbb{R} \rightarrow M$  originating at  $y$  at time 0, the point  $\gamma(l)$  is  $\epsilon$  close to  $x$ . For short we shall say that *all geodesics from  $y$  in time  $l$  focus within  $\epsilon$  from  $x$* .

It immediately follows that every  $Y_l^x$ -manifold is a  $Y^x$ -manifold, while every  $Y^x$ -manifold is a  $\tilde{Y}^x$ -manifold.

**2.3. Remark** (possible reformulations of Definitions 2.1 and 2.2). Our Corollary 3.3 says that if the requirements described in Definitions 2.1 and 2.2 are satisfied for all sufficiently small  $\epsilon > 0$ , then in fact they are satisfied for all  $\epsilon > 0$ . Thus in both definitions one can forget the condition that the requirement is supposed to be satisfied only for sufficiently small  $\epsilon$  and this would not change the class of manifolds described in the definition.

A much easier fact is that the requirement that the condition should be satisfied for all sufficiently small  $\epsilon > 0$ , can be substituted by the condition that there exists a sequence  $\{\epsilon_n\}_{n=1}^{\infty}$  of positive numbers with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  for which the condition is satisfied.

Indeed if the condition is true for all positive  $\epsilon < \bar{\epsilon}$ , then it is also satisfied for all the members of any such sequence  $\{\epsilon_n\}_{n=1}^{\infty}$  with all  $\epsilon_n < \bar{\epsilon}$ . On the other side, if  $r$  is the radius of a geodesically convex normal neighborhood of  $x$ , then for every positive  $\epsilon < \frac{r}{2}$  for which the condition is satisfied the corresponding  $l(\epsilon) > \frac{r}{2} > \epsilon$ . Without the loss of generality we can assume that all the sequence members are less than  $\frac{r}{2}$ . Then one chooses  $\bar{\epsilon} = \epsilon_1$ . Now given any positive  $\epsilon < \bar{\epsilon}$  choose  $K \in \mathbb{N}$  so that  $\epsilon_K < \epsilon$  and observe that if we put  $l(\epsilon) = l(\epsilon_K)$  (and  $y(\epsilon) = y(\epsilon_K)$  if we talk about  $\tilde{Y}^x$ -manifolds), then the desired condition is satisfied.

**2.4. Remark.** We do not know examples of  $\tilde{Y}^x$ -manifolds that are not  $Y^x$ -manifolds. Similarly we do not know examples of  $Y^x$ -manifolds that are not  $Y_l^x$ -manifolds for some nonzero  $l$ . It may be that these three classes actually coincide, but an attempt to prove this runs into a problem we describe below.

Given a Riemannian  $\tilde{Y}^x$ -manifold  $(M, g)$  let  $\{\epsilon_n\}_{n=1}^\infty$  be a positive sequence with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Then there is a sequence  $\{l_n\}_{n=1}^\infty$  of positive numbers and a sequence  $\{y_n\}_{n=1}^\infty$  of points in  $M$  such that for every geodesic  $\gamma : \mathbb{R} \rightarrow M$  parametrized by arc length and originating at  $y_n$ , the point  $\gamma(l_n)$  is  $\epsilon_n$  close to  $x$ .

Our Lemma 6.2 says that if  $(\tilde{y}, \tilde{l})$  is a limit point of the set  $\{(y_n, l_n)\}_{n=1}^\infty \subset M \times \mathbb{R}$ , then  $\tilde{l} > 0$  and  $(M, g)$  is a  $Y_{2\tilde{l}}^x$ -manifold.

Even though our Theorem 2.5 says that  $M$  has to be compact, it is not clear if one can always choose  $l_n$  so that they form a bounded sequence, and hence it is not clear whether the subset  $\{(y_n, l_n)\}_{n=1}^\infty \subset M \times \mathbb{R}$  necessarily has a limit point. This difficulty seems to be similar to [10, Question 7.70] discussed in Remark 1.3.

Our main result is a counterpart of the Bérard-Bergery Theorem.

**2.5. Theorem.** *Let  $M$  be a manifold of dimension at least 2 such that there exists a complete Riemannian metric  $g$  on  $M$  and a point  $x \in M$  with the property that  $(M^m, g)$  is a  $\tilde{Y}^x$ -manifold. Then  $M$  is a closed manifold and  $|\pi_1(M)| < \infty$ .*

Since every  $Y_l^x$ -manifold is a  $\tilde{Y}^x$ -manifold, Theorem 2.5 implies the first two out of three properties of  $Y_l^x$ -manifolds in the Bérard-Bergery Theorem. On the other hand, the proof of Theorem 2.5 contained in Section 6 is completely different from that of the Bérard-Bergery Theorem. It is based on Lorentzian geometry and notion of refocussing Lorentzian space-times introduced by Low [22, 23].

For reader's convenience in Section 4 we review necessary notions of Lorentz geometry. In Section 7 we discuss facts related to refocussing and open problems.

### 3. COROLLARIES OF THEOREM 2.5 AND OTHER RESULTS

**3.1. Fact.** *Let  $M^m, m = 2, 3$  be a closed manifold with finite fundamental group, then there is a complete Riemannian metric  $g$  on  $M$  and a point  $x \in M$  such that  $(M, g)$  is a  $Y_{2\pi}^x$ -manifold.*

*Proof.* By the Thurston Elliptization Conjecture [29] proved by Perelman [26, 27], a closed manifold  $M$  of dimension 3 with finite fundamental group is a quotient of the standard unit sphere  $S^3$  by a finite group of isometries. Thus  $M$  inherits the quotient metric  $g$  from the standard metric on the unit sphere. Clearly  $(M, g)$  is a  $Y_{2\pi}^x$ -manifold. The proof of Fact 3.1 in the case  $m = 2$  is similar to (but simpler than) the proof in the case  $m = 3$ .  $\square$

**3.2. Corollary.** *Let  $M^m, m = 2, 3$  be a manifold, such that there exists a complete Riemannian metric  $g$  on  $M$  and a point  $x \in M$  with the property that  $(M^m, g)$  is a  $\tilde{Y}^x$ -manifold. Then the ring  $H^*(M, \mathbb{Q})$  is generated by one element. Moreover there exists a complete Riemannian metric  $\tilde{g}$  on  $M$  such that  $(M, \tilde{g})$  is a  $Y_{2\pi}^x$ -manifold.*

*Proof.* By Theorem 2.5 the manifold  $M$  is closed and  $|\pi_1(M)| < \infty$ . Hence Corollary 3.2 follows from Fact 3.1 and Theorem 1.2.  $\square$

If the condition in Definitions 2.1 and 2.2 is actually satisfied for all  $\epsilon$ , then it is also satisfied for all sufficiently small  $\epsilon$ . The converse is also true, but requires

some thinking, even though this condition is automatically satisfied for all  $\epsilon > \text{diam}(M, g)$ .

**3.3. Corollary.** *If the condition in Definitions 2.1 and 2.2 of  $Y^x$ - and  $\tilde{Y}^x$ -manifolds is satisfied for all sufficiently small  $\epsilon > 0$ , then in fact it is satisfied for all  $\epsilon > 0$ .*

*Proof.* We give the proof for the  $\tilde{Y}^x$ -manifolds. The proof for  $Y^x$ -manifolds is similar and in fact simpler.

The case  $\dim M = 1$  is trivial. Assume  $(M^m, g), m > 1$  is a  $\tilde{Y}^x$ -manifold and let  $\bar{\epsilon} > 0$  be as in the definition of a  $\tilde{Y}^x$ -manifold. Choose a sequence of positive numbers  $\{\epsilon_n\}_{n=1}^\infty$ ,  $\epsilon_n < \bar{\epsilon}_n$  with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Choose a sequence  $\{l_n\}_{n=1}^\infty$  of positive numbers  $l_n > \epsilon_n$  and a sequence  $\{y_n\}_{n=1}^\infty$  of points in  $M$  such that all geodesics from  $y_n$  in time  $l_n$  focus within  $\epsilon_n$  from  $x$ .

$M$  is compact by Theorem 2.5. If the sequence  $\{l_n\}_{n=1}^\infty$  is bounded, then  $\{(y_n, l_n)\}_{n=1}^\infty$  contains a subsequence converging to some  $(\tilde{y}, \tilde{l})$ . Lemma 6.2 says that  $\tilde{l} > 0$  and  $(M, g)$  is a  $Y_{2\tilde{l}}^x$ -manifold. Then for a given  $\epsilon > 0$  one takes  $k \in \mathbb{N}$  so that  $2k\tilde{l} > \epsilon$  and puts  $y = x$  and  $l(\epsilon) = 2k\tilde{l}$ . Thus the condition of Definition 2.2 is in fact satisfied for all  $\epsilon > 0$ , rather than just for sufficiently small  $\epsilon$ .

If the sequence  $\{l_n\}_{n=1}^\infty$  is not bounded, then we choose a monotonically increasing subsequence  $\{l_{n_k}\}_{k=1}^\infty$  with  $\lim_{k \rightarrow \infty} l_{n_k} = +\infty$ . Now take any  $\epsilon > 0$  that is not necessarily less than  $\bar{\epsilon}$ . Choose  $K$  such that  $l_{n_K} > \epsilon$  and  $\epsilon_{n_K} < \epsilon$ . Clearly the point  $y = y_{n_K}$  and the positive number  $l = l_{n_K}$  satisfy the requirements of definition 2.2 for the chosen  $\epsilon > 0$ .  $\square$

**3.4. Corollary.** *Let  $(M, g)$  be a Riemannian manifold, then the possibly empty set  $\tilde{Z} = \{z \in M \mid (M, g) \text{ is a } \tilde{Y}^z\}$  is a closed subset of  $M$ .*

We do not know if for a Riemannian manifold  $(M, g)$ , the possibly empty set  $Z = \{z \in M \mid (M, g) \text{ is a } Y^z\}$  is always a closed subset of  $M$ .

*Proof.* If a connected  $M$  has dimension one, then the statement is obvious, since  $M$  is either diffeomorphic to  $S^1$  or to  $\mathbb{R}^1$ . In the first case  $\tilde{Z} = M$ , in the second case  $\tilde{Z} = \emptyset$ . Similarly the statement is obvious if  $\tilde{Z} = \emptyset$ . So we consider the case  $\tilde{Z} \neq \emptyset$ , and  $\dim M > 1$ .

$M$  is compact by Theorem 2.5. Thus there exists  $L > 0$  such that for every  $p \in M$  the exponential map restricted to the radius  $L$  ball centered at  $\mathbf{0} \in T_p M$  is a diffeomorphism. Take  $p \in \tilde{Z}$  and  $\bar{\epsilon}(p) > 0$  from the definition of a  $\tilde{Y}^p$ -manifold. For each  $\epsilon(p) < \bar{\epsilon}(p)$  we should have  $y(p, \epsilon) \in M$  and  $l(p, \epsilon) > \epsilon(p)$  such that if  $\gamma(t)$  is a unit speed geodesic satisfying  $\gamma(0) = y(p, \epsilon)$  then the distance  $d_g(\gamma(l(p, \epsilon)), p) < \epsilon$ . By definition of  $L$  we have that  $l(p, \epsilon) > \frac{L}{2}$ . Thus we can put  $\bar{\epsilon}(p) = \frac{L}{2}$  where the right hand side does not depend on  $p \in \tilde{Z}$ .

It suffices to show that if  $\hat{z}$  is a limit point of  $\tilde{Z}$ , then  $\hat{z} \in \tilde{Z}$ . Put  $\bar{\epsilon}(\hat{z}) = \frac{L}{2}$  and take any  $\epsilon < \bar{\epsilon}(\hat{z})$ . Take  $z_N \in \tilde{Z}$  such that  $d_g(\hat{z}, z_N) < \frac{\epsilon}{2}$ . Since  $z_N \in \tilde{Z}$  and  $\frac{\epsilon}{2} < \frac{L}{4} < \frac{L}{2}$  by the previous discussion there exist  $y \in M$  and  $l > \frac{L}{2}$  with the property that if  $\gamma(t)$  is a unit speed geodesic satisfying  $\gamma(0) = y$  then  $d_g(\gamma(l), z_N) < \frac{\epsilon}{2}$ . Since  $d_g(\hat{z}, z_N) < \frac{\epsilon}{2}$  we have  $d_g(\gamma(l), \hat{z}) < \epsilon$  for all unit speed geodesics  $\gamma(t)$  satisfying  $\gamma(0) = y$ .  $\square$

**3.5. Corollary.** *Let  $x \in \tilde{M}$  be a point in the connected total space of a Riemannian covering  $\rho: (\tilde{M}^m, \tilde{g}) \rightarrow (M^m, g), m \geq 2$ . Then the following two statements hold:*

- 1:**  $(\widetilde{M}, \widetilde{g})$  is a  $\widetilde{Y}^x$ -manifold if and only if  $(M, g)$  is a  $\widetilde{Y}^{\rho(x)}$ -manifold.  
**2:**  $(\widetilde{M}, \widetilde{g})$  is a  $Y^x$ -manifold if and only if  $(M, g)$  is a  $Y^{\rho(x)}$ -manifold.

The statement of this Theorem is false when  $M = S^1$  and  $\widetilde{M} = \mathbb{R}^1$ . Note that we use Theorem 2.5 only in the proof of statement **2**.

*Proof.* We prove statement **1**. Assume that  $(\widetilde{M}, \widetilde{g})$  is a  $\widetilde{Y}^x$ -manifold. For a sufficiently small  $\epsilon > 0$  take  $y \in M$  and  $l > \epsilon$  such that all the geodesics from  $y$  in time  $l$  focus within  $\epsilon$  from  $x$ . Since  $\rho$  is a Riemannian covering, we get that all the geodesics from  $\rho(y)$  in time  $l$  focus within  $\epsilon$  from  $\rho(x)$ . Thus  $(M, g)$  is a  $\widetilde{Y}^{\rho(x)}$ -manifold.

Now we prove the other implication. Take  $\bar{\epsilon} > 0$  as in the definition of  $(M, g)$  being a  $\widetilde{Y}^{\rho(x)}$ -manifold. We assume without the loss of generality that the exponential map  $\exp_{\rho(x)} : T_{\rho(x)}M \rightarrow M$  restricted to the ball of radius  $\bar{\epsilon}$  centered at  $\mathbf{0} \in T_{\rho(x)}M$  is a diffeomorphism. We put  $B$  to be the open neighborhood of  $\rho(x)$  that is the image of the restriction of  $\exp_{\rho(x)}$  map to this ball. Decreasing  $\bar{\epsilon}$  if necessary we can and do assume that  $B$  is trivially covered under  $\rho$ .

Choose any positive  $\epsilon < \bar{\epsilon}$  and let  $l > \epsilon$  and  $y \in M$  be such that all geodesics from  $y$  in time  $l$  focus within  $\epsilon$  from  $x$ . Choose a unit speed geodesic  $\gamma(t)$  such that  $\gamma(0) = y$ . Put  $z = \gamma(l) \in B$ . Put  $\tilde{B}$  to be the connected component of  $\rho^{-1}(B)$  containing  $x$  and put  $\tilde{z}$  to be the unique point of  $\rho^{-1}(z)$  located within  $\tilde{B}$ . Let  $\tilde{\gamma}(t)$  be the lift of the path  $\gamma(t)$  such that  $\tilde{\gamma}(l) = \tilde{z}$ . Put  $\tilde{y}$  to be  $\tilde{\gamma}(0)$ , so that  $\rho(\tilde{y}) = y$ . Since positive  $\epsilon < \bar{\epsilon}$  was arbitrary, to finish the proof it suffices to show that all the geodesics from  $\tilde{y}$  in time  $l$  focus within  $\epsilon$  from  $x$ . By our choice of  $y$  and  $l$  and since  $\rho$  is a covering, the end point of each of these geodesic arcs of length  $l$  starting from  $\tilde{y}$  is located within  $\epsilon$  from one of the points in  $\rho^{-1}(\rho(x))$  and thus within one of connected components of the preimage of  $B$  under  $\rho$ . These end points continuously depend on the initial directions of the geodesic arcs. Since  $\dim M > 1$ , the sphere of unit vectors in  $T_{\tilde{y}}\widetilde{M}$  is connected. Since  $B$  is covered trivially under  $\rho$ , the end points of all these length  $l$  geodesic arcs starting at  $\tilde{y}$  are located within epsilon from the same point in  $\rho^{-1}(\rho(x))$  as the end point  $\tilde{\gamma}(l)$ . This point is  $x$ .

Now we prove statement **2**. Clearly if  $(\widetilde{M}, \widetilde{g})$  is a  $Y^x$ -manifold, then  $(M, g)$  is a  $Y^{\rho(x)}$ -manifold.

Now we prove the other implication. Every  $(\widetilde{M}, \widetilde{g})$  is the base space of the Riemannian cover by the total space of the universal Riemannian cover of  $(M, g)$ . Thus by the previously proved implication it suffices to prove the statement when  $\rho$  is the universal Riemannian covering.

Let  $r > 0$  be such that  $\exp_{\rho(x)} : T_{\rho(x)}M \rightarrow M$  restricted to the radius  $r$  ball centered at  $\mathbf{0} \in T_{\rho(x)}M$  is a diffeomorphism. Let  $\bar{\epsilon} > 0$  be as in Definition 2.1. Choose any positive  $\epsilon < \min(\frac{r}{2}, \bar{\epsilon})$ . Take  $l_1$  such that all geodesics from  $\rho(x)$  in time  $l_1$  focus within  $\frac{\epsilon}{2}$  from  $\rho(x)$ . Similar to the proof of Corollary 3.4, we get that  $l_1 > \frac{r}{2}$ . As in the proof of the first statement of the Theorem we get that there is  $x_1 \in \rho^{-1}(\rho(x))$  such that all geodesics from  $x_1$  in time  $l_1$  focus within  $\frac{\epsilon}{2}$  from  $x$ . If  $x_1 = x$ , then we found the desired  $l = l_1$ .

Assume that  $x_1 \neq x$ . Let  $\Gamma$  be the group of deck transformations of the universal covering  $\rho$ . It acts transitively on  $\rho^{-1}(\rho(x))$  and we put  $\alpha_1 \in \Gamma$  to be such that  $x_1 = \alpha_1(x)$ . Since  $\rho$  is a Riemannian covering, we get that for each  $x' \in \rho^{-1}(\rho(x))$  all the geodesics from  $\alpha_1(x')$  in time  $l_1$  focus within  $\frac{\epsilon}{2}$  from  $x'$ .

Since the geodesic flow  $STM \times \mathbb{R} \rightarrow STM$  is continuous, we get that there is a small positive  $\tilde{\epsilon}_1 < \frac{\epsilon}{2}$ , such that for all the points  $y$  in the  $\tilde{\epsilon}_1$ -ball centered at  $x_1$  all the geodesics from  $y$  in time  $l_1$  focus within  $\epsilon = 2\frac{\epsilon}{2}$  from  $x$ .

Repeat the previous argument to find  $l_2 > 0$  and  $x_2 \in \rho^{-1}(\rho(x))$  such that all geodesics from  $x_2$  in time  $l_2$  focus within  $\tilde{\epsilon}_1$  of  $x_1$ . Thus all the geodesics from  $x_2$  in time  $l_2 + l_1$  focus within  $\epsilon$  of  $x$ . Put  $\alpha_2 \in \Gamma$  to be such that  $x_2 = \alpha_2(x_1)$ . Then for every  $x' \in \rho^{-1}(\rho(x))$  all the geodesics from  $\alpha_2\alpha_1(x')$  in time  $l_1 + l_2$  focus within  $\epsilon$  from  $x'$ ; all the geodesics from  $\alpha_2(x')$  in time  $l_2$  focus within  $\tilde{\epsilon}_1 < \epsilon$  of  $\alpha_1(x')$ ; and all the geodesics from  $\alpha_1(x')$  in time  $l_1$  focus within  $\frac{\epsilon}{2} < \epsilon$  of  $x'$ . Proceed by induction.

By Theorem 2.5  $|\rho^{-1}(\rho(x))| < \infty$ . So at a certain step of the inductive process the newly chosen  $x_j \in \rho^{-1}(\rho(x))$  will coincide with the previously chosen  $x_i, i < j$ . Then all the geodesics from  $x_i = x_j = \alpha_j\alpha_{j-1} \dots \alpha_{i+1}(x_i)$  in time  $l_j + l_{j-1} + \dots + l_{i+1}$  focus within  $\epsilon$  from  $x_i$ . Since  $\rho$  is a Riemannian cover, we get that all the geodesics from  $\alpha_j\alpha_{j-1} \dots \alpha_{i+1}(x)$  in time  $l_j + l_{j-1} + \dots + l_{i+1}$  focus within  $\epsilon$  from  $x$ .  $\square$

#### 4. BRIEF INTRODUCTION TO LORENTZIAN MANIFOLDS

A *Lorentzian manifold*  $(X^{m+1}, g^L)$  is a Pseudo-Riemannian manifold whose metric tensor  $g^L$  is of signature  $(m, 1)$ . In other words, each point  $p \in X$  of a Lorentzian manifold  $(X^{m+1}, g^L)$  has a coordinate neighborhood with coordinates  $(x_1, \dots, x_{m+1})$  such that the metric tensor  $g^L|_{T_p X \times T_p X}$  is of the form

$$dx_1^2 + dx_2^2 + \dots + dx_m^2 - dx_{m+1}^2,$$

where  $T_p X$  is the tangent space of  $X$  at  $p$ .

A nonzero vector  $v \in T_p X$  of a Lorentzian manifold  $(X^{m+1}, g^L)$  is said to be *timelike*, *non-spacelike*, *null* (*lightlike*), or *spacelike* if  $g^L(v, v)$  is negative, non-positive, zero, or positive, respectively. A piecewise smooth curve  $\gamma(t)$  is called *timelike*, *non-spacelike*, *null*, or *spacelike* if all of its velocity vectors  $\gamma'(t)$  are respectively timelike, non-spacelike, null, or spacelike.

For each  $p \in X$  the set of all non-spacelike vectors in  $T_p X$  has two connected components that are hemicones. A continuous (with respect to  $p \in X$ ) choice of a hemicone of non-spacelike vectors in  $T_p X$  is called a *time orientation* of  $(X, g^L)$ . The non-spacelike vectors from the chosen hemicones are called *future pointing vectors*. A piecewise smooth curve is said to be *future directed* if all of its velocity vectors are future pointing. A connected time-oriented Lorentzian manifold without boundary is called a *space-time*.

For two events  $x, y$  in a space-time  $(X, g^L)$  we write  $x \leq y$  if  $x = y$  or if there is a piecewise smooth future directed non-spacelike curve from  $x$  to  $y$ . For  $x \in (X, g^L)$ , the spaces

$$J^+(x) = \{y \in X | x \leq y\} \quad \text{and} \quad J^-(x) = \{y \in X | y \leq x\}$$

are called the *causal future* and *causal past of  $x$*  respectively. Two events  $x, y$  are *causally related* if  $y \in J^\pm(x)$ . A space-time  $(X^{m+1}, g^L)$  is *causal* if it does not have closed future directed non-spacelike curves.

An open set in a space-time is *causally convex* if there are no future directed non-spacelike curves intersecting it in a disconnected set. A space-time is *strongly causal* if every point in it has arbitrarily small causally convex neighborhoods. A

strongly causal space-time  $(X, g^L)$  is *globally hyperbolic* if  $J^+(x) \cap J^-(y)$  is compact for all  $x, y \in X$ .

A *Cauchy surface*  $M$  is a subset of a space-time  $(X, g^L)$  such that for every inextendible future directed non-spacelike curve  $\gamma(t)$  in  $X$  there exists exactly one value  $t_0$  of  $t$  with  $\gamma(t_0) \in M$ . A space-time can be shown to be globally hyperbolic if and only if it admits a Cauchy surface, see [16, pages 211-212].

**4.1. Useful Facts.** (a) Every Lorentzian manifold  $(X, g^L)$  has a unique Levi-Civita connection, see for example [4, page 22]. This allows one to talk about geodesics and about null-geodesics, i.e., geodesics whose velocity vector is null everywhere. An affine reparameterization of a null geodesic also is a null geodesic. However, contrary to the Riemannian geometry, null geodesics do not have a canonical parametrization. A curve is called a *pregeodesic* if it can be reparameterized to be a geodesic.

(b) The pioneer result of Geroch [15] says that every globally hyperbolic space-time  $(X^{m+1}, g)$  is *homeomorphic* to  $M^m \times \mathbb{R}$  where every  $M \times t \subset X$  is a Cauchy surface.

(c) Bernal and Sanchez [7, Theorem 1], [8, Theorem 1.1], [9, Theorem 1.2] have proved that a Cauchy surface  $M$  in a globally hyperbolic space-time  $(X^{m+1}, g^L)$  can always be chosen to be smooth and *spacelike* i.e.,  $g^L|_{TM}$  is Riemannian. Moreover in this case  $X$  is diffeomorphic to  $M \times \mathbb{R}$ , each slice  $M \times t$  is a smooth spacelike Cauchy surface, and any two such Cauchy surfaces are diffeomorphic. They also proved [6] that in the definition of globally hyperbolic space-times it suffices to require that  $(X^{m+1}, g^L)$  is causal rather than that it is strongly causal.

(d) Let  $(M, g)$  be a Riemannian manifold, and let  $f : (\alpha, \beta) \rightarrow (0, +\infty)$  be a smooth positive function, where  $-\infty \leq \alpha < \beta \leq +\infty$ . Then the warped product space-time  $(M \times (\alpha, \beta), f(t)g \oplus -dt^2)$  is globally hyperbolic and each  $M \times t$  is a smooth spacelike Cauchy surface, see [4, Theorem 3.66].

(e) Two Lorentz manifolds  $(X_1, g_1^L)$  and  $(X_2, g_2^L)$  are said to be *conformal equivalent* if there exists a diffeomorphism  $f : X_1 \rightarrow X_2$  and a positive smooth function  $\Omega : X_1 \rightarrow (0, +\infty)$  such that  $g_1^L = \Omega f^*(g_2^L)$ . If  $\gamma$  is a timelike or spacelike or null curve in  $(X_1, g_1^L)$ , then clearly  $f(\gamma)$  is respectively a timelike or spacelike or null curve in  $(X_2, g_2^L)$ . Moreover if  $\gamma$  is a null pregeodesic, then  $f(\gamma)$  also is a null pregeodesic [4, Lemma 9.17]. The similar statement is generally false for spacelike and timelike pregeodesics.

## 5. REFOCUSING AND EXAMPLES

**5.1. Definition** (Strongly refocussing Lorentzian manifolds). We say that a Lorentzian manifold  $(X^{m+1}, g^L)$  is *strongly refocussing* at  $x \in X$  if there exists  $y \in X$  such that for every (inextendible) null geodesic  $\nu(t)$  with  $\nu(0) = y$  there exists nonzero  $\tau = \tau(\nu)$  such that  $\nu(\tau) = x$ . Note that this  $\tau$  may and generally does depend on the choice of the null geodesic  $\nu$ .

We say that a Lorentzian manifold is *strongly refocussing* if it is strongly refocussing at some point.

We require  $\tau \neq 0$  since otherwise we always have refocussing via choosing  $y = x$  and  $\tau = 0$ . This definition means that all the light rays through  $y$  also pass through  $x$  (for nontrivial reasons).

**5.2. Definition** (Weakly refocussing Lorentzian manifolds). We say that  $(X^{m+1}, g^L)$  is *(weakly) refocussing* at  $x \in X$  if there exists open  $U \ni x$  such that given any

open  $V$  with  $x \in V \subset U$  there exists  $y \notin V$  such that all the null geodesics through  $y$  pass through  $V$ . Note that these null geodesics are not required to pass through  $x$ .

We say that a Lorentzian manifold is *weakly refocussing* if it is weakly refocussing at some point.

This definition was introduced by Low [22, 23] for the physically interesting strongly causal space-times.

**5.3. Remark.** Let  $(X_1, g_1^L)$  and  $(X_2, g_2^L)$  be conformal space-times. Let  $f : X_1 \rightarrow X_2$  be a diffeomorphism and  $\Omega : X_1 \rightarrow (0, +\infty)$  be a smooth positive function such that  $\Omega f^*(g_2^L) = g_1^L$ . If  $\gamma(t)$  is a null pregeodesic for  $(X_1, g_1^L)$ , then  $f(\gamma(t))$  is a null pregeodesic for  $(X_2, g_2^L)$ , see [4, Lemma 9.17].

Thus if  $(X_1, g_1^L)$  is refocussing (respectively strongly refocussing) at  $x \in X_1$ , then  $(X_2, g_2^L)$  is refocussing (respectively strongly refocussing) at  $f(x)$ . In particular if  $(X_1, g_1^L)$  and  $(X_2, g_2^L)$  are conformal equivalent, then one is refocussing exactly when the other one refocussing, and the same is true for strong refocussing.

**5.4. Example** (Chernov-Rudiyak construction [14] of strongly refocussing space-times). Let  $(M, g)$  be a  $Y_l^x$  manifold for some  $x \in M$  and nonzero  $l \in \mathbb{R}$ . Consider the Lorentzian product manifold  $(X^{m+1}, g^L) = (M \times \mathbb{R}, g \oplus -dt^2)$ . Then all the null geodesics through  $(x, t-l)$  pass through  $(x, t)$ . Thus the globally hyperbolic space-time  $(X^{m+1}, g^L)$  is strongly refocussing at  $(x, t)$  for each  $t \in \mathbb{R}$  (see [14, Section 11, Remark 7]).

Example 5.4 can be modified to yield a strongly refocussing Lorentzian manifold with a metric that is not a product metric. Indeed, let  $U$  be an open neighborhood of the singular hypersurface in  $X^{m+1}$  covered by the union of the arcs of the null geodesics from  $(x, -l)$  to  $(x, 0)$ . Let  $g_U^L$  be any Lorentzian metric that equals to  $g \oplus -dt^2$  on  $U$ . Then  $(M \times \mathbb{R}, g_U^L)$  is strongly refocussing at  $(x, 0)$ . This gives a vast collection of strongly refocussing Lorentzian manifolds with a metric that is not the product metric.

**5.5. Example** (Weakly refocussing space-times). From the proof of Theorem 2.5 it is easy to see that if  $(M, g)$  is a  $\tilde{Y}^x$ -manifold for some  $x \in M$ , then  $(M \times \mathbb{R}, g \oplus -dt^2)$  is refocussing at  $(x, t)$  for each  $t \in \mathbb{R}$ .

**5.6. Example** (Kinlaw [17] example of globally hyperbolic space-times that are refocussing but not strongly refocussing at a point). Let  $g$  be the standard metric on a unit sphere  $S^m \subset \mathbb{R}^{m+1}$ . Then  $(X_1, g_1^L) = (S^m \times (-\pi, \pi), g \oplus -dt^2)$  contains a codimension one submanifold  $\Sigma = \{(x, 0) | x \in S^m\}$ , such that  $(X_1, g_1^L)$  is refocussing but not strongly refocussing at each point of  $\Sigma$ . Note that  $(X_1, g_1^L)$  is strongly refocussing at  $(x, -\delta)$  for small  $\delta > 0$  and thus the globally hyperbolic manifold  $(X_1, g_1^L)$  is strongly refocussing.

It is easy to see that this example is neither spacelike, nor timelike, nor null geodesically complete. However by [4, Lemma 9.17] it is conformal equivalent to a globally hyperbolic space-time  $(X_2, g_2^L)$  that is null and timelike geodesically complete. By Remark 5.3 this  $(X_2, g_2^L)$  also has a hypersurface formed by points such that  $(X_2, g_2^L)$  is refocussing but not strongly refocussing at these points.

The following Theorem is close in spirit to our Corollary 3.4.



**5.7. Theorem** (Kinlaw [17]). *Let  $(X^{m+1}, g^L)$  be a strongly causal space-time. Then the possibly empty set  $\tilde{Z} = \{z \in X \mid (X, g^L) \text{ is refocussing at } z\}$  is a closed subset of  $X$ .*

Example 5.6 shows that the set  $Z = \{z \in X \mid (X, g^L) \text{ is strongly refocussing at } z\}$  does not have to be a closed subset of a strongly causal  $(X^{m+1}, g^L)$ .

## 6. PROOF OF THEOREM 2.5

Let  $(M, g)$  be a  $\tilde{Y}^x$ -manifold for some point  $x \in M$ . Let  $\text{pr} : STM \rightarrow M$  denote the tangent unit sphere bundle of  $M$ . The fiber of  $\text{pr}$  over a point  $y \in M$  is denoted by  $ST_y M$ . For  $v \in STM$ , let  $\gamma_v : \mathbb{R} \rightarrow STM$  be the unique unit speed geodesic with  $\dot{\gamma}_v(0) = v$ . There are smooth maps

$$p : STM \times \mathbb{R} \rightarrow M,$$

$$p : v \times \tau \mapsto \gamma_v(\tau)$$

and

$$q : STM \times \mathbb{R} \rightarrow STM,$$

$$q : v \times \tau \mapsto \dot{\gamma}_v(\tau).$$

We recall that there is a positive real number  $\bar{\epsilon}$  such that the property in the definition of  $\tilde{Y}^x$ -manifolds holds for all  $\epsilon$  with  $0 < \epsilon < \bar{\epsilon}$ . Let  $\{\epsilon_n\}_{n=1}^\infty$  be a sequence of positive numbers  $\epsilon_n < \bar{\epsilon}$  with  $\lim \epsilon_n = 0$ . Since  $(M, g)$  is a  $\tilde{Y}^x$ -manifold, there exist sequences of positive real numbers  $\{l_n\}_{n=1}^\infty$ ,  $l_n > \epsilon_n$  and points  $\{y_n\}_{n=1}^\infty$  such that

$$\text{Im}(p|_{ST_{y_n} M \times l_n}) \subset B(x, \epsilon_n),$$

where  $B(x, \epsilon_n)$  denotes the open ball in  $M$  about  $x$  of radius  $\epsilon_n$ .

To begin with let us assume that the sequence  $\{(y_n, l_n)\}_{n=1}^\infty$  of points in  $M \times \mathbb{R}$  has no convergent subsequence. Then, as we show in Lemma 6.1, the globally hyperbolic Lorentzian product manifold  $(X, g^L) = (M \times \mathbb{R}, g \oplus -dt^2)$  is refocussing. By a result of Low, its Cauchy surface  $M \times 0 = M$  is a closed manifold, see [22, 23], [14, Section 11, Proposition 6]. On the other hand, the result of Rudyak and the first author says that the fundamental group of the Cauchy surface  $M \times 0$  has to be finite, see [14, Theorem 15]. This completes the proof of Theorem 2.5 under the assumption that the sequence  $\{(y_n, l_n)\}$  has no convergent subsequence.

Suppose now that the sequence  $\{(y_n, l_n)\}$  has a convergent subsequence. Then, by Lemma 6.2 below, the manifold  $(M, g)$  is a  $Y_l^x$ -manifold for some  $l$ . Hence, in this case, the conclusion of Theorem 2.5 immediately follows from Lemma 6.1 and the Bérard-Bergery Theorem 1.2.

In the rest of the section we prove Lemmas 6.1 and 6.2.

**6.1. Lemma.** *Suppose that the sequence  $\{(y_n, l_n)\}$  does not have a convergent subsequence. Then the globally hyperbolic Lorentzian product manifold  $(X, g^L) = (M \times \mathbb{R}, g \oplus -dt^2)$  is refocussing at  $(x, 0)$ .*

*Proof.* Put  $g^R = g \oplus dt^2$  to be the product Riemannian metric on  $M \times \mathbb{R}$  and put

$$U = B((x, 0), \bar{\epsilon}) = \{(y, t) \in M \times \mathbb{R} \mid d_{g^R}((y, t), (x, 0)) < \bar{\epsilon}\}$$

to be the open ball neighborhood of  $(x, 0) \in M \times \mathbb{R}$  of radius  $\bar{\epsilon}$ . Let  $V \subset U$  be any neighborhood of  $(x, 0)$ . Put

$$\tilde{V} = \{y \in M \mid y \times 0 \in V\}$$

to be the open neighborhood of  $x$  and put  $\epsilon > 0$  with  $\bar{\epsilon} > \epsilon$  to be such that

$$B(x, \epsilon) = \{y | d_g(x, y) < \epsilon\} \subset \tilde{V}.$$

Since the sequence  $\{(y_n, l_n)\}$  does not have a convergent subsequence and

$$\lim_{n \rightarrow \infty} \epsilon_n = 0,$$

there exists a positive integer  $N$  such that  $\epsilon_N < \epsilon$  and  $(y_N, l_N) \notin U$ . If  $w \in T_{(y, \tau)}(M \times \mathbb{R})$  is a null vector with components

$$(w_M, w_{\mathbb{R}}) \in T_y M \oplus T_{\tau} \mathbb{R} = T_{(y, \tau)}(M \times \mathbb{R}),$$

then  $g(w_M, w_M) = -w_{\mathbb{R}} \cdot w_{\mathbb{R}}$ , where  $\cdot$  is the standard Riemannian metric on  $\mathbb{R}^1$ . Since  $g \oplus -dt^2$  is a Lorentzian product metric, the geodesics in  $(M \times \mathbb{R}, g \oplus -dt^2)$  should project to geodesics in  $(M, g)$  and in  $(\mathbb{R}, -dt^2)$ . Thus all the null geodesics through  $(y_N, -l_N) \notin V$  intersect  $\tilde{V} \times 0 \subset V$  and hence they all intersect  $V$ . Hence  $(M \times \mathbb{R}, g \oplus -dt^2)$  is refocussing.  $\square$

**6.2. Lemma.** *Suppose that the sequence of points*

$$\{(y_n, l_n)\}_{n=1}^{\infty} \subset M \times \mathbb{R}$$

*contains a subsequence  $\{(y_{n_k}, l_{n_k})\}_{k=1}^{\infty}$  converging to a point  $(\tilde{y}, \tilde{l})$ . Then  $\tilde{l} \neq 0$ , and  $(M, g)$  is a  $Y_{2\tilde{l}}^x$ -manifold.*

*Proof.* Put  $r > 0$  to be such that the exponential map  $\exp_x : T_x M \rightarrow M$  restricted to the radius  $r$  ball centered at  $\mathbf{0} \in T_x M$  is a diffeomorphism. Then each  $l_n > r$  and thus the limit value  $\tilde{l}$  is non-zero.

Without loss of generality we can assume that each point  $y_{n_k}$  belongs to a prescribed geodesically convex neighborhood  $W$  of  $\tilde{y}$ . For  $v \in ST_{\tilde{y}} M$  let  $v_{n_k} \in ST_{y_{n_k}} M$  denote the vector obtained by the parallel transport of  $v$  along the unique geodesic in  $W$  connecting  $\tilde{y}$  to  $y_{n_k}$ . Then

$$\lim_{k \rightarrow \infty} (v_{n_k}, l_{n_k}) = (v, \tilde{l}).$$

Since the map  $p$  is continuous, its values  $p(v_{n_k}, l_{n_k})$  converge to  $p(v, \tilde{l})$  as  $k \rightarrow \infty$ . On the other hand, each point  $p(v_{n_k}, l_{n_k})$  is  $\epsilon_{n_k}$ -close to the point  $x$ . In view of the convergence  $\epsilon_{n_k} \rightarrow 0$ , we conclude that  $p(v, \tilde{l}) = x$ . Consequently,

$$\text{Im } p|_{ST_{\tilde{y}} M \times \tilde{l}} = x,$$

i.e.,  $\gamma(\tilde{l}) = x$  for each geodesic  $\gamma$  emitted from  $\tilde{y}$  with  $|\dot{\gamma}(0)| = 1$ . Thus

$$q|_{ST_{\tilde{y}} M \times \tilde{l}} : ST_{\tilde{y}} M = S^{m-1} \rightarrow ST_x M = S^{m-1}$$

is a smooth embedding and hence a diffeomorphism for dimensional reasons. Consequently,

$$\text{Im } p|_{ST_x M \times \tilde{l}} = \tilde{y}.$$

This implies that  $(M, g)$  is a  $Y_{2\tilde{l}}^x$ -manifold.  $\square$

**6.3. Remark.** As it has been explained in Example 5.4, one deduces that the globally hyperbolic Lorentzian product manifold  $(X, g^L) = (M \times \mathbb{R}, g \oplus -dt^2)$  is refocussing at  $(x, 0)$  (and hence at each  $(x, t)$  with  $t \in \mathbb{R}$  for the reason of symmetry).

**6.4. Remark.** Theorem 2.5 can be also proved so that its proof is independent of the Bérard-Bergery Theorem, which we used in the proof of Theorem 2.5 under the hypothesis of Lemma 6.2. Indeed, suppose that the sequence of points  $\{(y_n, l_n)\}$  contains a sub-sequence converging to a point  $(\tilde{y}, \tilde{l})$ . Since the limit value  $\tilde{l}$  is different from zero, we may still apply the argument of Lemma 6.1 to complete the proof of Theorem 2.5. On the other hand, the statement of Lemma 6.2 is somewhat stronger than what we can deduce using the argument in Lemma 6.1 as it asserts that all geodesics emitted from  $\tilde{y}$  return precisely to the point  $\tilde{y}$  at the moment  $2\tilde{l}$ .

## 7. INTRIGUING FACTS RELATED TO $Y^x$ - AND $\tilde{Y}^x$ -RIEMANNIAN MANIFOLDS, REFOCUSING SPACE-TIMES, POSITIVE LEGENDRIAN ISOTOPY AND OPEN QUESTIONS

**7.1.  $Y_l^x$ -Riemannian manifolds, causality in space-times, Low Conjecture and the Legendrian Low Conjecture.** Low Conjecture [18], [19], [20], [21] and the Legendrian Low conjecture due to Nataro and Tod [24] reformulate causality in a globally hyperbolic space-time  $(X^{m+1}, g^L)$  in terms of link theory. Basically they ask if it is true that when the Cauchy surface is diffeomorphic to an open subset of  $\mathbb{R}^2$  or of  $\mathbb{R}^3$ , then two events  $x, y \in X$  are causally related if and only if the spheres of null geodesics passing through  $x$  and  $y$  are linked (in the appropriate sense) in the contact manifold of all non-parameterized future pointing null geodesics in  $(X^{m+1}, g)$ . This motivated a problem communicated by Penrose on Arnold's problem lists [2, Problem 8], [3, Problem 1998-21].

Stefan Nemirovski and the first author [12, Theorem A, Theorem C] proved the Low and the Legendrian Low conjectures. They also proved [13, Theorem 10.4] that the statements of these conjectures remain true for all globally hyperbolic space-times  $(X^{m+1}, g^L)$ ,  $m > 1$  such that the total space of the universal cover of its Cauchy surface  $M^m$  is an open manifold.

If  $(M, g)$  is a  $Y_l^x$  Riemannian manifold, then these conjectures are false in the strongly refocussing  $(M \times \mathbb{R}, g \oplus -dt^2)$ , see [13, Example 10.5]. In the physically most interesting case of a  $(3+1)$ -dimensional globally hyperbolic space-time  $(X^{3+1}, g^L)$  we get that if the Legendrian Low conjecture fails for  $(X^{3+1}, g^L)$ , then the Cauchy surface of  $X$  admits a Riemannian metric making it into a  $Y_l^x$ -manifold, see [12, page 1322].

**7.2. Topology of a refocussing globally hyperbolic space-time.** An interesting question is what should be the topology of a Cauchy surface  $M$  of a refocussing globally hyperbolic  $(X^{m+1}, g^L)$ . Low [22, 23] proved that  $M$  has to be a closed manifold, see also [14, Section 11, Proposition 6]. Rudyak and the first author proved that the universal Lorentzian cover of a refocussing globally hyperbolic space-time  $(X^{m+1}, g^L)$ ,  $m > 1$  is a refocussing globally hyperbolic space-time, see [14, Theorem 14]. Thus  $|\pi_1(M)| < \infty$ .

It is interesting to know if the third implication of the Bérard-Bergery Theorem holds for a Cauchy surface  $M$  of a globally hyperbolic refocussing space-time  $(X^{m+1}, g)$ ,  $m > 1$ , i.e., is it true that the ring  $H^*(M^m, \mathbb{Q})$  is generated by one element? This is true for  $\dim M = 2, 3$ . Indeed Fact 3.1 says that such  $M$  admits a Riemannian metric  $g_q$  making  $(M, g_q)$  into a  $Y_{2\pi}^x$ -manifold. Now Bérard-Bergery Theorem [5], [10, Theorem 7.37, page 192] says that the ring  $H^*(M, \mathbb{Q})$  is generated by one element.

Paul Kinlaw [17] noticed that this discussion implies the following intriguing observation: if a globally hyperbolic space-time  $(X^{m+1}, g^L)$ ,  $m = 2, 3$  is refocussing, then it admits a globally hyperbolic Lorentzian metric  $\tilde{g}^L = g_q \oplus -dt^2$  such that  $(X^{m+1}, \tilde{g}^L)$  is strongly refocussing.

We do not know examples of globally hyperbolic space-times that are refocussing but not strongly refocussing. However Example 5.6 shows that the situation is quite nontrivial.

**7.3.  $\tilde{Y}^x$ - and  $Y^x$ -manifolds and positive Legendrian isotopy.** Since  $(M, g)$  is a Riemannian manifold we have the natural identification  $STM = ST^*M$ . The spherical cotangent bundle  $ST^*M$  has a natural contact structure and the  $S^{m-1}$ -fiber  $S_x$  of  $ST^*M \rightarrow M$  over a point  $x \in M$  is a Legendrian submanifold. For each  $t$  the map  $q|_{ST^*M \times t} : ST^*M \rightarrow ST^*M$  preserves the contact structure and hence it maps Legendrian submanifolds to Legendrian submanifolds. Moreover the map  $\phi : S_x \times [0, \infty) \rightarrow ST^*M$  defined by  $\phi(z, t) = q(z, t)$  is a positive Legendrian isotopy, i.e. it is a Legendrian isotopy such that the evaluation of the contact form on the velocity vectors of the trajectory curves  $\phi_z(t) = \phi(z, t) : [0, \infty) \rightarrow ST^*M$ ,  $z \in S_x$  is positive.

If  $(M, g)$  is a  $Y_t^x$ -manifold, then  $\phi : S_x \times [0, l] \rightarrow ST^*M$  is a positive Legendrian isotopy of the fiber  $S_x$  to itself. If a Cauchy surface  $M^m$ ,  $m > 1$  of a globally hyperbolic space-time  $(X^{m+1}, g)$  is such that there is no positive Legendrian isotopy of an  $S^{m-1}$ -fiber of  $ST^*M$  to itself, then the Legendrian Low conjecture holds for  $(X^{m+1}, g)$ , see [12, Section 7] and [13, proof of Theorem 10.4]. In [13, Corollary 8.1] Nemirovski and the first author proved that if  $ST^*M$  admits a positive Legendrian isotopy of  $S_x$  to itself, then  $M$  is compact and has finite  $\pi_1(M)$ , i.e. the universal cover of  $M$  is compact. In particular this gives yet another proof of the first two statements of Bérard-Bergery Theorem 1.2. A question in [13, Example 8.3] asks whether the existence of a positive Legendrian isotopy of  $S_x$  to  $S_x$  implies that the rational cohomology ring  $H^*(M, \mathbb{Q})$  is generated by one element.

It may be that the result of [13] can be strengthened to show that if the universal cover of  $M$  is not compact, then given two not necessarily distinct points  $x, y \in M$  and a sufficiently small neighborhood  $U$  of  $S_x$  there is no positive Legendrian isotopy  $\phi : S_y \times [0, 1] \rightarrow ST^*M$  such that  $\text{Im}(\phi(S_y \times \frac{1}{2})) \cap U = \emptyset$  and  $\text{Im}(\phi(S_y \times 1)) \subset U$ . If such a result holds it would give another proof of our Theorem 2.5. One can also ask the question whether the existence of such a positive Legendrian isotopy implies that the ring  $H^*(M, \mathbb{Q})$  is generated by one element.

If  $(M, g)$ ,  $m = 2, 3$  is a  $\tilde{Y}^x$ - or a  $Y^x$ -manifold then the ring  $H^*(M, \mathbb{Q})$  is generated by one element, see Corollary 3.2. So one can ask if it holds in all dimensions. This question does not seem to be immediately reducible to the question in [13, Example 8.3].

Indeed given a  $Y^x$ -manifold  $(M, g)$  and a sequence  $\{\epsilon_n\}_{n=1}^\infty$  of sufficiently small positive numbers converging to zero, put  $\{l_n\}_{n=1}^\infty$ ,  $l_n > \epsilon_n$  to be a sequence as in Definition 2.1. Put  $B(x, \epsilon_n)$  be the metric ball of radius  $\epsilon_n$ -centered at  $x$ . Clearly  $\text{Im} p|_{S_x \times l_n}$  is a Legendrian submanifold of  $ST^*B(x, \epsilon_n) \subset ST^*M$  that can be obtained from  $S_x$  by a positive Legendrian isotopy within  $ST^*M$ . However it is not clear if this submanifold can be deformed to  $S_x$  by a positive Legendrian isotopy inside  $ST^*B(x, \epsilon_n)$  or even inside  $ST^*M$ . A similar difficulty arises for  $\tilde{Y}^x$ -manifolds.

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